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THE SWING: PARAMETRIC RESONANCE†

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The instability of oscillations of a weightless rod with a concentrated mass, sliding periodically along the rod axis is investigated. This is the simplest model of a child's swing. The amplitude of the displacement of the mass and viscous friction, due to the air resistance, are assumed to small, while the periodic excitation function is arbitrary. Asymptotic formulae for the regions of instability (parametric resonance) in three-dimensional space of the system parameters, corresponding to swinging of the swing, are obtained and investigated. Examples are given.

The problem of the swing is undoubtedly one of the classical problems in mechanics. It is well known that, to swing a swing one must crouch in the middle vertical position and straighten up in the extreme positions, i.e. perform oscillations with a frequency which is approximately double the frequency of natural oscillations of the swing. While swinging, to maintain the oscillations one can crouch half as often. Despite the popularity of the swing, in the literature on oscillations and stability [1–14], where this problem is referred to, there are no general formulae for describing the regions of instability, which explain the phenomenon of the swinging of a swing. The problem of the instability of a swing is solved below using an approach which involves the use of the derivatives of the monodromy matrix with respect to the parameters [15–18]. The method of solving the problem is rigorous and is based on an analysis of the behaviour of Floquet multipliers. The results of this paper were briefly described in [19].

1. BASIC RELATIONS

The simplest model of a swing is described by the oscillations of a weightless rod with a concentrated mass, which slides periodically along the rod axis, this scheme describes the oscillations of a pendulum of variable length, which varies periodically with time (Fig. 1). The amplitude of the displacement of the mass or the change in the pendulum length are assumed to be small. The small viscous friction due to the air resistance is taken into account.

The equation of motion of a swing (a pendulum of variable length) [1, 3] is derived using the theorem of the change in the angular momentum and taking viscous friction into account and has the form

$$(ml^2\dot{\theta})^{\cdot}+\gamma l^2\dot{\theta}+mgl\sin\theta = 0 \tag{1.1}$$

where m is the mass, l is the length, θ is the angle of deflection of the pendulum from the vertical, γ is the coefficient of viscous friction, due to the air resistance, and g is the acceleration due to gravity. The dot denotes a derivative with respect to time t. It is assumed that the pendulum length varies as follows:

$$l = l_0 + a\varphi(\Omega t), \quad \int_0^{2\pi}\varphi(\tau)d\tau = 0 \tag{1.2}$$

where l_0 is the mean length of the pendulum, a and Ω are the amplitude and frequency of the excitation, respectively, and $\varphi(\tau)$ is an arbitrary continuous periodic function of period 2π with a mean value of

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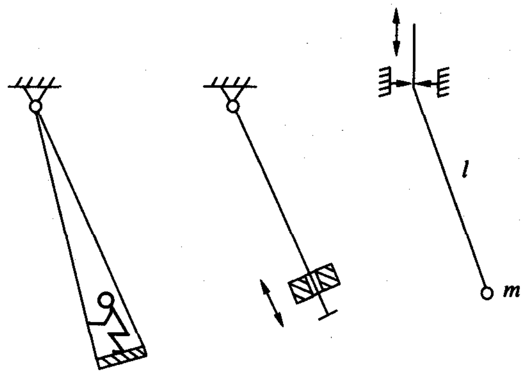

Fig. 1

zero. The amplitude a and the friction coefficient γ are assumed to be small. It is required to determine for what values of the parameters the trivial equilibrium position of the system $\theta = 0$ becomes unstable which leads to swinging of the swing.

We will introduce the following dimensionless variables and parameters

$$\tau = \Omega t, \quad \varepsilon = \frac{a}{l_0}, \quad \beta = \frac{\gamma}{m}\sqrt{\frac{l_0}{g}}, \quad \omega = \frac{1}{\Omega}\sqrt{\frac{g}{l_0}}, \quad x_1 = \theta, \quad x_2 = \frac{l^2\dot{\theta}}{l_0^2\Omega} \tag{1.3}$$

Then Eq. (1.1) can be written in the form of a system of first-order equations

$$\frac{dx_1}{d\tau} = \left(\frac{l_0}{l}\right)^2 x_2, \quad \frac{dx_2}{d\tau} = -\omega^2\frac{l}{l_0}\sin x_1 - \beta\omega x_2; \quad \frac{l}{l_0} = 1 + \varepsilon\varphi(\tau) \tag{1.4}$$

In these variables the requirement that the periodic function $\varphi(\tau)$ should be continuous can be relaxed, assuming it to be only piecewise-continuous. Similar variables were used in [20] to solve an optimal control problem.

The right-hand sides of the first two equations of (1.4) are non-linear functions of the vector $\mathbf{x} = (x_1, x_1)$ and are periodic in τ with period 2π. Equations (1.4) depend explicitly on three independent parameters ω, ε and β, where the last two are assumed to be small

$$\varepsilon \ll 1, \quad \beta \ll 1 \tag{1.5}$$

It is required to obtain the regions of instability of the trivial solution $\mathbf{x} = 0$ (parametric resonance) in the three-dimensional space of the parameters $\mathbf{p} = (\varepsilon, \beta, \omega)$.

If the problem of a swing is considered for the periodic function $\varphi(\tau_0 + \tau)$, where τ_0 is some phase shift, this problem of instability is equivalent to the initial problem, which can be shown by making the time conversion $\tau' = \tau_0 + \tau$. Hence, the phase shift of the periodic function is unimportant for the fact of the instability of the motion, but it has a considerable effect on the swinging time [20].

2. SOLUTION OF THE PROBLEM OF INSTABILITY

By Lyapunov's theorem one can judge the stability and instability of a non-linear system (1.4) from the linear approximation [21]. Linearization of this system leads to the equations

$$\dot{\mathbf{x}} = \mathbf{G}\mathbf{x} \tag{2.1}$$

$$\mathbf{G} = \left\| \begin{matrix} 0 & [1 + \varepsilon\varphi(\tau)]^{-2} \\ -\omega^2[1 + \varepsilon\varphi(\tau)] & -\beta\omega \end{matrix} \right\| \tag{2.2}$$

The fundamental matrix $\mathbf{X}(t)$ of system (2.1) is found from the matrix differential equation with initial conditions

$$\dot{\mathbf{X}} = \mathbf{G}\mathbf{X}, \quad \mathbf{X}(0) = \mathbf{I} \tag{2.3}$$

where **I** is the identity matrix, and is called the matricant. The monodromy (Floquet) matrix is defined by the equality $\mathbf{F} = \mathbf{X}(T)$ [21, 22]. To investigate the stability of the linear system (2.1), (2.2) we will use Floquet's theory, according to which a linear system with periodic coefficients is stable if all the eigenvalues ρ (multipliers) of the monodromy matrix **F** are less than unity in modulus, and unstable if at least one of the multipliers is greater than unity in modulus.

Suppose we know the monodromy matrix $\mathbf{F}_0 = \mathbf{F}(\mathbf{p}_0)$ for a certain n-dimensional vector of the parameters $\mathbf{p}_0$. We give the vector of the parameters an increment in the form $\mathbf{p} = \mathbf{p}_0 + \Delta\mathbf{p}$, as a result of which the matrix **G** and, consequently, the matricant $\mathbf{X}(t)$ obtain increments also. This correspondingly leads to a change in the monodromy matrix **F**. Expressions have been obtained in [15, 16] for the first and second derivatives of the monodromy matrix with respect to the parameters in the form of integrals over a period

$$\frac{\partial \mathbf{F}}{\partial p_k} = \mathbf{F}_0 \int_0^T \mathbf{H}_k(\tau)d\tau \tag{2.4}$$

$$\frac{\partial^2 \mathbf{F}}{\partial p_i \partial p_j} = \mathbf{F}_0\left[\int_0^T \mathbf{H}_{ij}(\tau)d\tau + \int_0^T \mathbf{H}_i(\tau)\left(\int_0^\tau \mathbf{H}_j(\zeta)d\zeta\right)d\tau + \int_0^T \mathbf{H}_j(\tau)\left(\int_0^\tau \mathbf{H}_i(\zeta)d\zeta\right)d\tau\right] \tag{2.5}$$

where

$$\mathbf{H}_k(\tau) = \mathbf{X}_0^{-1}(\tau)\frac{\partial \mathbf{G}}{\partial p_k}(\mathbf{p}_0, \tau)\mathbf{X}_0(\tau),$$

$$\mathbf{H}_{ij}(\tau) = \mathbf{X}_0^{-1}(\tau)\frac{\partial^2 \mathbf{G}}{\partial p_i \partial p_j}(\mathbf{p}_0, \tau)\mathbf{X}_0(\tau), \quad i, j, k = 1, \ldots, n$$

The zero subscript denotes that the corresponding quantity is taken at $\mathbf{p} = \mathbf{p}_0$.

Note that, to calculate the derivatives (2.4) and (2.5) it is only necessary to know the matricant $\mathbf{X}_0(t)$ and the derivatives of the matrix **G** with respect to the parameters, calculated at $\mathbf{p} = \mathbf{p}_0$. Using the derivatives (2.4) and (2.5), we can write the increment of the monodromy matrix in the form

$$\mathbf{F}(\mathbf{p}_0 + \Delta\mathbf{p}) = \mathbf{F}_0 + \sum_{k=1}^{n} \frac{\partial \mathbf{F}}{\partial p_k}\Delta p_k + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 \mathbf{F}}{\partial p_i \partial p_j}\Delta p_i \Delta p_j + \ldots \tag{2.6}$$

Knowing the derivatives of the monodromy matrix we can calculate the values of this matrix in the neighbourhood of the point $\mathbf{p}_0$ and, consequently, estimate the behaviour of the multipliers (the eigenvalues of the monodromy matrix **F**), responsible for the stability of system (2.1) when the parameters change.

If we put $\varepsilon = 0$ and $\beta = 0$ in relations (2.1) and (2.2), from Eqs (2.3) it is easy to obtain the matricant and the matrix inverse to it

$$\mathbf{X}_0(t) = \begin{Vmatrix} \cos\omega t & \omega^{-1}\sin\omega t \\ -\omega\sin\omega t & \cos\omega t \end{Vmatrix}, \quad \mathbf{X}_0^{-1}(t) = \begin{Vmatrix} \cos\omega t & -\omega^{-1}\sin\omega t \\ \omega\sin\omega t & \cos\omega t \end{Vmatrix} \tag{2.7}$$

Hence, when $\varepsilon = 0$ and $\beta = 0$ the monodromy matrix has the form

$$\mathbf{F}_0 = \mathbf{X}_0(2\pi) \tag{2.8}$$

The eigenvalues of this matrix (the multipliers) are

$$\rho_{1,2} = \cos 2\pi\omega \pm i \sin 2\pi\omega \tag{2.9}$$

For all values of $\omega \neq k/2$ ($k = 1, 2, \ldots$) the multipliers are complex-conjugate quantities and lie on the unit circle (stability). For small changes in the parameters ε, β and ω in the neighbourhood of the point $\mathbf{p}_0 = (0, 0, \omega)$, $\omega \neq k/2$ ($k = 1, 2, \ldots$) by virtue of the continuity the multipliers remain complex-conjugate quantities. We then have the following quadratic equation for the multipliers

$$\rho^2 + A\rho + B = 0 \tag{2.10}$$

where the free term, by Liouville's formula [21], is described by the expression

$$B = \exp\left(\int_0^{2\pi} \mathrm{tr}\,\mathbf{G}\,dt\right) = \exp(-2\pi\beta\omega) \tag{2.11}$$

Since, by Vieta's theorem, from relations (2.10) and (2.11) where $\beta > 0$ we have

$$\rho_1\rho_2 = B < 1 \tag{2.12}$$

for the complex-conjugate multipliers, it follows from inequality (2.12) that $|\rho_{1,2}| < 1$. Hence, a small change in the parameters ε, β and ω, when $\beta > 0$, in the neighbourhood of the point $\mathbf{p}_0 = (0, 0, \omega)$, $\omega \neq k/2$ shifts the multipliers inside the unit circle, which indicates asymptotic stability.

Consequently, instability (parametric resonance) can only arise in the neighbourhood of the points

$$\mathbf{p}_0\colon \varepsilon = 0, \quad \beta = 0, \quad \omega = k/2, \quad k = 1, 2\ldots \tag{2.13}$$

in which the multipliers are doubled: $\rho_1 = \rho_2 = \cos\pi k$.

To obtain the regions of parametric resonance we expand the monodromy matrix $\mathbf{F}$ in the neighbourhood of the point $\mathbf{p}_0$ in a Taylor series in the parameters ε, β and $\Delta\omega = \omega - k/2$:

$$\mathbf{F}(\mathbf{p}) = \mathbf{F}(\mathbf{p}_0) + \frac{\partial\mathbf{F}}{\partial\varepsilon}\varepsilon + \frac{\partial\mathbf{F}}{\partial\beta}\beta + \frac{\partial\mathbf{F}}{\partial\omega}\Delta\omega + \ldots \tag{2.14}$$

From formulae (2.4), using relations (2.2), (2.7) and (2.8), we calculate the values of the derivatives $\partial\mathbf{F}/\partial\varepsilon$, $\partial\mathbf{F}/\partial\beta$ and $\partial\mathbf{F}/\partial\omega$ at $\mathbf{p} = \mathbf{p}_0$. As a result, we have from Eq. (2.6), apart from first-order infinitesimal terms,

$$\mathbf{F}(\mathbf{p}) = \cos\pi k \begin{Vmatrix} 1 + \frac{3}{4}k\pi b_k\varepsilon - \frac{1}{2}k\pi\beta & \frac{4}{k}\pi\Delta\omega - \frac{3}{2}\pi a_k\varepsilon \\ -k\pi\Delta\omega - \frac{3}{8}k^2\pi a_k\varepsilon & 1 - \frac{3}{4}k\pi b_k\varepsilon - \frac{1}{2}k\pi\beta \end{Vmatrix} \tag{2.15}$$

Here we have introduced the Fourier coefficients of the function $\varphi(\tau)$

$$a_k = \frac{1}{\pi}\int_0^{2\pi} \varphi(\tau)\cos k\tau d\tau, \quad b_k = \frac{1}{\pi}\int_0^{2\pi} \varphi(\tau)\sin k\tau d\tau, \quad k = 1, 2\ldots \tag{2.16}$$

For the matrix (2.15) we obtain the approximate values of the multipliers

$$\rho_{1,2} = (-1)^k(1 - k\pi\beta/2) \pm \pi\sqrt{D} \tag{2.17}$$

$$D = k^2 r_k^2 \varepsilon^2 - (2\Delta\omega)^2, \quad r_k = \frac{3}{4}\sqrt{a_k^2 + b_k^2} \tag{2.18}$$

The system is unstable if at least one of the multipliers is greater in modulus than unity [21, 22]. This condition is satisfied when $\beta < 0$ and the system is unstable, while when $\beta \geq 0$ this condition is satisfied only when $\sqrt{D} > \beta k/2$. Hence, taking expression (2.18) into account, we obtain that the instability region of (parametric resonance) lies inside the half of the cone

$$k^2\beta^2/4 + 4(\omega - k/2)^2 < k^2 r_k^2 \varepsilon^2, \quad \beta \geq 0 \tag{2.19}$$

which is connected with the half space $\beta < 0$ (Fig. 2). The instability regions are shown hatched. Inequality (2.19) can also be represented in the somewhat more convenient form

$$(\beta/2)^2 + (2\omega/k - 1)^2 < r_k^2 \varepsilon^2, \quad \beta \geq 0 \tag{2.20}$$

Note that formulae (2.19) and (2.20) are approximations of the first order for the instability regions. It follows from them, in particular, that the kth resonance region in the first approximation depends only on the kth Fourier coefficients of the periodic excitation function.

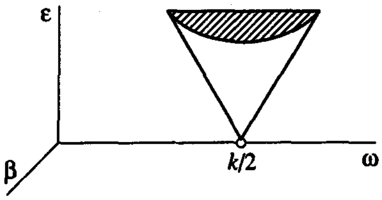

Fig. 2

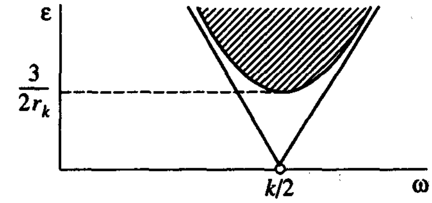

Fig. 3

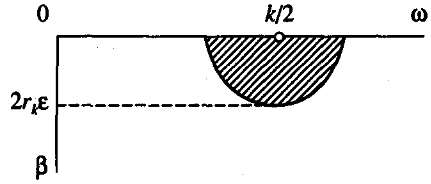

Fig. 4

Putting $\beta = 0$ in inequality (2.19), we obtain the zones of parametric resonance when there is no friction

$$-kr_k/2 < \varepsilon^{-1}(\omega - k/2) < kr_k/2 \tag{2.21}$$

The section of the cone (2.19) by the plane $\beta = \text{const} \geq 0$ gives the zone of parametric resonance, bounded by a hyperbola (Fig. 3). The asymptotes of this hyperbola can be found from inequalities (2.21). When there is friction ($\beta > 0$) the minimum amplitude of the excitation of resonance, by inequality (2.19), is

$$\varepsilon_{\min} = \beta/(2r_k) \tag{2.22}$$

The section of the region (2.19) by the plane $\varepsilon = \text{const}$ represents half an ellipse with semiaxes $|\omega - k/2| = kr_k\varepsilon/2$ and $\beta = 2r_k\varepsilon$ (Fig. 4). Note that as the friction coefficient β increases, the width of the resonance zone narrows with respect to the frequency ω and disappears when $\beta > 2r_k\varepsilon$.

We will analyse the evolution of the resonance regions as the resonance number k increases. It is well known that if a periodic function $\varphi(\tau)$ is continuous together with its sth order derivatives, then for the Fourier coefficients a_k and b_k we have the relations $a_k k^{s+1} \to 0$, $b_k k^{s+1} \to 0$ as $k \to \infty$. Hence, for continuously differentiable functions, the quantities kr_k tend to zero as $k \to \infty$. This indicates that the cone (2.19) narrows as k increases. Hence, it also follows that for fixed β and as k increases, the minimum amplitude of the excitation of resonance (2.22) increases without limit. This explains the fact that it is easier for a swing to swing at lower resonances $k = 1, 2$, whereas for higher values of k greater excitation amplitudes and forces are required to attain resonance.

Reverting to dimensional quantities, we obtain from relations (1.3) and (2.13) that swinging occurs at excitation frequencies Ω close to the critical values

$$\Omega_k = \frac{2}{k}\sqrt{\frac{g}{l_0}}, \quad k = 1, 2, \ldots \tag{2.23}$$

Note that $\sqrt{g/l_0}$ is the frequency of natural oscillations of a pendulum of mean length l_0. The corresponding regions of resonance, by formula (2.20), are described by the inequalities

$$\frac{\gamma^2 l_0}{4gm^2} + \left(\frac{\Omega}{\Omega_k} - 1\right)^2 < \frac{r_k^2 a^2}{l_0^2} \tag{2.24}$$

similar to the inequalities which describe the regions of instability of a pendulum with a vertically oscillating suspension point [17]. The difference is that the right-hand sides of the inequalities depend on the resonance number k. This difference can obviously be explained by the fact that an acceleration, proportional to the square of the excitation frequency occurs in the equation of the oscillations of a pendulum with an oscillating suspension point.

3. THE DEGENERATE CASE

When $a_k = 0$ and $b_k = 0$ we have $r_k = 0$ and the first-order approximations (2.19) and (2.20) degenerate into a straight line $\beta = 0$, $\omega = k/2$. In this case, it is necessary to use higher-order approximations to obtain the resonance regions more accurately.

In view of the degeneracy of the linear terms in ε in formulae (2.14) and (2.15), in this case we have the following approximate expression for the monodromy matrix

$$\mathbf{F}(\mathbf{p}) = \mathbf{F}(\mathbf{p}_0) + \frac{1}{2}\frac{\partial^2 \mathbf{F}}{\partial \varepsilon^2}\varepsilon^2 + \frac{\partial \mathbf{F}}{\partial \beta}\beta + \frac{\partial \mathbf{F}}{\partial \omega}\Delta\omega + \dots \tag{3.1}$$

The point $\mathbf{p}_0$ is given by formula (2.13). Carrying out the calculations using formula (2.5) and also relations (2.2), (2.7) and (2.8), we obtain

$$\mathbf{F}(\mathbf{p}) = \cos\pi k \begin{Vmatrix} 1 + \frac{1}{2}k\pi\xi_k\varepsilon^2 - \frac{1}{2}k\pi\beta & \frac{4}{k}\pi\Delta\omega + \pi(\zeta_k + \eta_k)\varepsilon^2 \\ -k\pi\Delta\omega + \frac{1}{4}k^2\pi(\zeta_k - \eta_k)\varepsilon^2 & 1 - \frac{1}{2}k\pi\xi_k\varepsilon^2 - \frac{1}{2}k\pi\beta \end{Vmatrix} \tag{3.2}$$

In this formula we have used the following notation for the coefficients

$$\begin{aligned} \xi_k &= -\frac{3}{2\pi}\int_0^{2\pi}\varphi^2(t)\sin kt\,dt + \frac{3k}{4\pi}\int_0^{2\pi}\varphi(t)\int_0^t\varphi(\tau)\cos k\tau\,d\tau\,dt \\ \zeta_k &= \frac{3}{2\pi}\int_0^{2\pi}\varphi^2(t)\cos kt\,dt + \frac{3k}{4\pi}\int_0^{2\pi}\varphi(t)\int_0^t\varphi(\tau)\sin k\tau\,d\tau\,dt \\ \eta_k &= \frac{3}{2\pi}\int_0^{2\pi}\varphi^2(t)\,dt - \frac{9k}{4\pi}\int_0^{2\pi}\varphi(t)\sin kt\int_0^t\varphi(\tau)\cos k\tau\,d\tau\,dt \end{aligned} \tag{3.3}$$

The system is unstable if at least one of the eigenvalues (multipliers) of the matrix (3.2) is greater than unity in modulus. Carrying out calculations similar to the previous ones, we obtain that the instability region of (parametric resonance) in the degenerate case $a_k = 0$, $b_k = 0$, is given by the inequalities.

$$\beta^2 + 4\left(\frac{2\omega}{k} - 1 + \frac{\eta_k\varepsilon^2}{2}\right)^2 < R_k^2\varepsilon^4, \quad \beta \geq 0 \tag{3.4}$$

$$R_k = \sqrt{\xi_k^2 + \zeta_k^2} \tag{3.5}$$

Note that when $\eta_k \neq 0$ the resonance regions (3.4), unlike (2.19), are not symmetrical about the plane $\omega = k/2$. If $R_k > |\eta_k|$, the resonance regions lie on both sides of the plane $\omega = k/2$, while when $R_k \leq |\eta_k|$ they lie on one side of this plane.

The section of the region (3.4) by the plane $\beta = \text{const} \geq 0$ gives a resonance zone bounded by a generalized hyperbola. In Fig. 5 we show the case $R_k > |\eta_k|$, when the resonance zones (shown hatched) lie on both sides of the vertical $\omega = k/2$. From inequality (3.4) we obtain directly the minimum value of the amplitude for which parametric resonance occurs

$$\varepsilon_{\min} = \sqrt{\beta/R_k}$$

For continuous functions $\varphi(\tau)$ this quantity tends to infinity when k increases without limit.

When there is no friction ($\beta = 0$) we obtain from formula (3.4) inequalities which confine the resonance frequency between two parabolas (Fig. 5)

$$\omega_k^- < \omega < \omega_k^+; \quad \omega_k^\pm = \frac{k}{2} \pm \frac{\varepsilon^2 k}{4}(R_k \mp \eta_k) \tag{3.6}$$

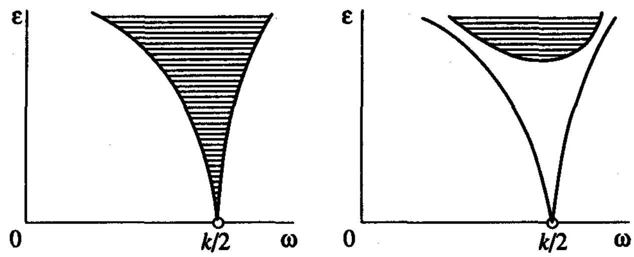

Fig. 5

In dimensional quantities, we obtain from inequalities (3.4), taking relations (1.3) into account,

$$\frac{\gamma^2 l_0}{4gm^2}+\left(\frac{\Omega}{\Omega_k}-1+\frac{\eta_k}{2}\left(\frac{a}{l_0}\right)^2\right)^2<\frac{R_k^2}{4}\left(\frac{a}{l_0}\right)^4 \tag{3.7}$$

The critical frequencies Ω_k are given by formula (2.23). Correspondingly, for $\gamma = 0$ we have the resonance zones

$$\chi_k^-<\Omega<\chi_k^+;\quad \chi_k^\pm=\Omega_k\left(1\pm\frac{1}{2}\left(\frac{a}{l_0}\right)^2(R_k\mp\eta_k)\right) \tag{3.8}$$

4. EXAMPLES

We will consider, as the first example, the excitation of oscillations using a periodic piecewise-constant function [5, 12]

$$\varphi(\tau)=\begin{cases}1, & 0\le\tau\le\pi\\ -1, & \pi<\tau<2\pi\end{cases} \tag{4.1}$$

For this function we have

$$\begin{aligned} &a_{2k-1}=0,\quad b_{2k-1}=\frac{4}{\pi(2k-1)},\quad r_{2k-1}=\frac{3}{\pi(2k-1)}\\ &a_{2k}=b_{2k}=r_{2k}=0;\quad k=1,2,\ldots\end{aligned} \tag{4.2}$$

Hence, by inequalities (2.24), all the odd regions are described by the formula

$$\frac{\gamma^2 l_0}{4gm^2}+\left(\frac{\Omega}{\Omega_{2k-1}}-1\right)^2<\frac{9a^2}{\pi^2 l_0^2(2k-1)^2},\quad k=1,2,\ldots \tag{4.3}$$

while the even resonance regions are degenerate in the first approximation. It can be seen directly from this formula how rapidly the cone of instability contracts as k increases.

If we put $\gamma = 0$ in (4.3), we obtain a formula for the resonance zones of the system without friction

$$\Omega_k^-<\Omega<\Omega_k^+;\quad \Omega_k^\pm=\frac{2}{2k-1}\sqrt{\frac{g}{l_0}}\left(1\pm\frac{3a}{\pi l_0(2k-1)}\right) \tag{4.4}$$

When $k = 1$ this result is identical with that obtained previously ([5, formulae (4.74)–(4.76)], if we put $\Omega = 2\omega_0 + \Delta\Omega$ in (47.4) and use the expansion tg $(\pi\omega_0/(2\Omega)) \approx 1 + \pi\Delta\Omega/(4\omega_0)$; see also the results in [6], which have a more complex form).

For even resonances, using formulae (3.3) we calculate $\xi_{2k} = 0$, $\zeta_{2k} = 3/2$, $\eta_{2k} = 3/4$, $R_{2k} = 3/2$ and, by inequalities (3.7), we obtain the instability regions

$$\frac{\gamma^2 l_0}{4gm^2}+\left(\frac{\Omega}{\Omega_{2k}}-1+\frac{3}{8}\left(\frac{a}{l_0}\right)^2\right)^2<\frac{9}{16}\left(\frac{a}{l_0}\right)^4,\quad k=1,2,\ldots \tag{4.5}$$

It is interesting to note that the resonance region remains unchanged for all critical frequencies Ω_{2k}.

When there is no friction ($\gamma = 0$), from formulae (3.8) and (2.23) we obtain a relation for the resonance zones

$$\frac{1}{k}\sqrt{\frac{g}{l_0}}\left(1 - \frac{9}{8}\left(\frac{a}{l_0}\right)^2\right) < \Omega < \frac{1}{k}\sqrt{\frac{g}{l_0}}\left(1 + \frac{3}{8}\left(\frac{a}{l_0}\right)^2\right) \tag{4.6}$$

As the second example we will take the periodic function in the form $\varphi(\tau) = \cos\tau - \sin 2\tau$. Then

$$a_1 = 1, \quad b_2 = -1, \quad a_2 = b_1 = 0, \quad r_1 = r_2 = 3/4, \quad a_k = b_k = r_k = 0, \quad k = 3, 4, \dots$$

Hence, all the resonance regions are degenerate, apart from the first and second.
By relations (2.23) and (2.24), we have for the first resonance region

$$\frac{\gamma^2 l_0}{4gm^2} + \left(\frac{\Omega}{2\sqrt{g/l_0}} - 1\right)^2 < \frac{9a^2}{16l_0^2} \tag{4.7}$$

Hence, when there is no friction ($\gamma = 0$) we obtain a relation for the first resonance zone

$$2\sqrt{\frac{g}{l_0}}\left(1 - \frac{3a}{4l_0}\right) < \Omega < 2\sqrt{\frac{g}{l_0}}\left(1 + \frac{3a}{4l_0}\right) \tag{4.8}$$

Similar relations are also obtained for the second resonance. They correspond to inequalities (4.7) and (4.8) when $2\sqrt{g/l_0}$ is replaced by $\sqrt{g/l_0}$.
The regions obtained coincide (in the first approximation) with the corresponding resonance regions for the case of the functions $\varphi(\tau) = \cos\tau$ and $\varphi(\tau) = \sin 2\tau$.

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